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REARRANGEMENT INVARIANT SUBSPACES OF LORENTZ FUNCTION SPACES

BY

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ABSTRACT

The main result is that for $2 \leqq q \leqq p < \infty$ the only subspaces of the Lorentz function space $L_{pq}[0,1]$ which are isomorphic to r.i. function spaces on $[0,1]$ are, up to equivalent renormings, $L_{pq}[0,1]$ and $L_2[0,1]$.

1. Introduction

In this paper we prove that the only subspaces of the classical Lorentz function spaces $L_{pq}[0,1]$, $2 \leqq q \leqq p < \infty$, which are isomorphic to rearrangement invariant function spaces on $[0,1]$ are, up to equivalent renormings, $L_{pq}[0,1]$ and $L_2[0,1]$. This extends a result of W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri [7] which states that for $2 \leqq p < \infty$ the only r.i. subspaces of $L_p[0,1]$ are, up to equivalent renormings, $L_p[0,1]$ and $L_2[0,1]$. In fact, we actually prove a similar result for a much larger class of Lorentz function spaces which includes the Lorentz–Zygmund spaces $L_{pq;\alpha}$, $2 \leqq q < p < \infty$, $0 \leqq \alpha < \infty$ [3]. Our main result is Theorem 1 which states that for regular, submultiplicative weights $w(x)$ and $2 \leqq p < \infty$, the only r.i. subspaces of the Lorentz function space $L_{w,p}[0,1]$ are, up to equivalent renormings, $L_{w,p}[0,1]$ and $L_2[0,1]$. Theorem 2 gives the converse, that is, if $w(x)$ is regular, $1 < p < \infty$, and if $L_{w,p}[0,1]$ and $L_2[0,1]$ are the only r.i. subspaces of $L_{w,p}[0,1]$, then $w(x)$ is submultiplicative. That submultiplicativity is the proper condition to place on the weight $w(x)$ is suggested by the analogous statement for Orlicz function spaces from [7], and also by similar results of Z. Altshuler, P. G. Cassaza and B. L. Lin [2] on Lorentz sequence spaces. In the final section an example is given which indicates the difficulty in classifying subspaces of $L_{w,p}[0,1]$ when $w(x)$ is not submultiplicative.

Our notation is standard and follows that of [8] and [7]. We denote by $|A|$ the Lebesgue measure of a measurable set A of $\mathbf{R}$. If f is a measurable function we denote by d_f the distribution function of $|f|$, that is,

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$$d_f(t) = |\{s : |f(s)| > t\}|,$$

and we denote by f^* the decreasing rearrangement of $|f|$, that is,

$$f^*(t) = \inf\{s > 0 : d_f(s) \leqq t\}.$$

Let $w(x)$ be a non-increasing, strictly positive function on $I = (0,1]$ or $(0,\infty)$ for which $\int_0^1 w(t)dt = 1$, and let $S(x) = \int_0^x w(t)dt$. Then for $1 \leqq p < \infty$ the norm of a function f in $L_{w,p}(I)$ is defined by

$$\|f\| = \left(\int_I f^*(t)^p w(t)dt\right)^{1/p}.$$

A simple integration-by-parts argument shows that we may also write

$$\|f\| = \left(\int_0^\infty S(d_f(t))d(t^p)\right)^{1/p}.$$

The weight $w(x)$ is called *regular* if $\inf_x S(2x)/S(x) > 1$, and *submultiplicative* if there is a constant $C < \infty$ such that $w(xy) \leqq Cw(x)w(y)$ for all x, y.

Note that for $w(t) = (q/p)t^{q/p-1}$, $1 \leqq q \leqq p < \infty$, we have $L_{w,q}(I) \equiv L_{pq}(I)$, and for $w(t) = c(p,q,\alpha)\cdot t^{q/p-1}\cdot(1+|\log t|)^{\alpha q}$, $1 \leqq q < p < \infty$, $0 \leqq \alpha < \infty$, we have $L_{w,q}(I) \equiv L_{pq;\alpha}(I)$, where $c(p,q,\alpha)$ is a constant chosen to satisfy $\int_0^1 w(t)dt = 1$. Also note that in each of these cases $w(x)$ is both regular and submultiplicative.

It is easy to see that $L_{w,p}$ is p-convex with constant 1 for any weight $w(x)$ and any $1 \leqq p < \infty$; the following theorem, which is a synthesis of known results, gives necessary and sufficient conditions for $L_{w,p}$ to be q-concave for some $q < \infty$:

THEOREM ([6], [1]). *For $1 < p < \infty$, the following are equivalent*:

(i) *$L_{w,p}$ is uniformly convex,*

(ii) *$L_{w,p}$ is q-concave for some $q < \infty$,*

(iii) *$w(x)$ is regular,*

(iv) *there exists a constant $C < \infty$ such that $xw(x) \leqq S(x) \leqq Cxw(x)$ for all x.*

Consequently [8, theorem 2.c.6] for $p > 1$ the Haar system $(h_{n,i})_{n=0,}^{\infty}\,_{i=1}^{2^n}$, defined by $h_{0,1} \equiv 1$ and for $n \geqq 1$ by

$$h_{n,i}(t) = \begin{cases} 1 & \text{if } t \in [(2i-2)2^{-n-1}, (2i-1)2^{-n-1}), \\ -1 & \text{if } t \in [(2i-1)2^{-n-1}, 2i\cdot 2^{-n-1}), \\ 0 & \text{otherwise,} \end{cases}$$

forms an unconditional basis for $L_{w,p}[0,1]$ exactly when $w(x)$ is regular.

We will also make use of the dilation operators D_s, $0 < s < \infty$. If $I = [0,\infty)$ these operators are defined by $(D_s f)(t) = f(t/s)$, $0 \leqq t < \infty$, and if $I = [0,1]$ we define

$$(D_s f)(t) = \begin{cases} f(t/s) & \text{if } 0 \leqq t \leqq s, \\ 0 & \text{if } s < t \quad (\text{in case } s < 1), \end{cases}$$

for $0 \leqq t \leqq 1$. Note that if $I = [0,\infty)$ or $I = [0,1]$ and $s \leqq 1$, the distribution of $D_s f$ is $s \cdot d_f$. Indeed,

$$\begin{aligned} |\{t : |D_s f|(t) > a\}| &= |\{t : |f(t/s)| > a\}| \\ &= s \cdot |\{t : |f(t)| > a\}|. \end{aligned}$$

2. The proof of our main result requires the use of several deep results from [7]. Of particular importance are theorem 2.1 and theorem 6.1 of [7] which we state below:

THE CLASSIFICATION FORMULA. *For every $M \geqq 1$, $C \geqq 1$ and every integer $m \geqq 1$ there exists a constant $D = D(M, C, m) < \infty$ such that if $(y_i)_{i=1}^n$ is a finite M-symmetric normalized basic sequence in a Banach lattice Y which is 2-convex and $2m$-concave with both constants $\leqq C$ then, for every choice of scalars $(a_i)_{i=1}^n$,*

$$\begin{aligned} D^{-1} \cdot \left\| \sum_{i=1}^n a_i y_i \right\| &\leqq \max \left\{ \left(\sum_\pi \left\| \max_{1 \leqq i \leqq n} |a_{\pi(i)} y_i| \right\|^{2m} \Big/ n! \right)^{1/2m}, \left\| \sum_{i=1}^n y_i \right\| \cdot \left(\sum_{i=1}^n |a_i|^2/n \right)^{1/2} \right\} \\ &\leqq D \cdot \left\| \sum_{i=1}^n a_i y_i \right\|, \end{aligned}$$

where Σ_π refers to summation over all permutations π of $\{1, \cdots, n\}$.

THE CLASSIFICATION THEOREM. *Let X be a* r.i. *function space on $[0,1]$ for which the Haar system is an unconditional basis. Let Y be a* r.i. *function space on $[0,1]$ or $[0,\infty)$ which does not contain uniformly isomorphic copies of l_∞^n for all n. If X embeds isomorphically into Y then one of the following three (non-exclusive) possibilities holds:*

(i) *There exists a constant $C < \infty$ such that*

$$\|f\|_Y \leqq C \|f\|_X$$

for every $f \in X$.

(ii) *The Haar system in X is equivalent to a sequence of disjointly supported functions in Y.*

(iii) *X is equal to* $L_2[0,1]$ *or* $L_2[0,\infty)$ *up to an equivalent renorming.*

The proof of Theorem 1 begins with an application of the Classification Theorem; however, Case (ii) may be excluded when $Y = L_{w,p}[0,1]$ for $w(x)$ regular and $p \geqq 2$. For $p > 2$ this is easy to see. Indeed, by [5], every disjoint normalized sequence in $L_{w,p}[0,1]$ has a subsequence equivalent to the unit vector basis of l_p. In particular, $L_{w,p}[0,1]$ $(p>2)$ cannot contain a disjoint sequence equivalent to the unit vector basis of l_2. For $p = 2$ we need the following lemma which was first noticed by G. Schechtman:

Lemma 1. *Let X be a* r.i. *function space on* $[0,1]$ *for which the Haar system is an unconditional basis. If the Haar basis in X is equivalent to a sequence of disjoint functions in* $L_{w,2}[0,1]$, *then X is equal to* $L_2[0,1]$ *up to an equivalent renorming.*

Proof. Suppose that $(h_{n,i})$, the Haar basis in X, is C-equivalent to a disjointly supported sequence $(f_{n,i})$ in $L_{w,2}[0,1]$. We first show that there is an infinite subset $M \subset \mathbf{N}$ so that

$$(*) \qquad \left\| \sum_{n\in M}\sum_{i=1}^{2^n} a_{n,i}h_{n,i}\right\|_X \overset{D}{\sim} \left(\sum_{n\in M}\left\|\sum_{i=1}^{2^n} a_{n,i}h_{n,i}\right\|_X^2\right)^{1/2},$$

for every choice of scalars $(a_{n,i})_{n\in M, i=1}^{2^n}$, where $D = 2C^2$. To see this, choose inductively a subsequence (n_k) and a sequence $\varepsilon_k \searrow 0$ which for $k \geqq 1$ satisfy:

(1) $\sum_{i=1}^{2^{n_k}} |\operatorname{supp} f_{n_k,i}| < \varepsilon_k$, and

(2) if $|A| < \varepsilon_{k+1}$, then $\|\chi_A \cdot f\| < \frac{1}{2}\|f\|$ for $f \in [f_{n_k,i}]_{i=1}^{2^{n_k}}$.

Set $M = \{n_k : k \geqq 1\}$. Then, for any scalars $(a_{n,i})_{n\in M, i=1}^{2^n}$, we have

$$\begin{aligned}\left\|\sum_{n\in M}\sum_{i=1}^{2^n} a_{n,i}h_{n,i}\right\|_X^2 &\geqq C^{-2}\cdot\left\|\sum_{n\in M}\sum_{i=1}^{2^n} a_{n,i}f_{n,i}\right\|^2 \\ &\geqq C^{-2}\cdot\sum_{k=1}^{\infty}\int_{\varepsilon_{k+1}}^{\varepsilon_k}\left(\sum_{i=1}^{2^{n_k}} a_{n_k,i}f_{n_k,i}\right)^{*2}(t)w(t)dt \\ &\geqq \frac{3}{4}C^{-2}\sum_{n\in M}\left\|\sum_{i=1}^{2^n} a_{n,i}f_{n,i}\right\|^2 \\ &\geqq \frac{3}{4}C^{-4}\cdot\sum_{n\in M}\left\|\sum_{i=1}^{2^n} a_{n,i}h_{n,i}\right\|_X^2,\end{aligned}$$

and since $L_{w,2}[0,1]$ is 2-convex,

$$\left\| \sum_{n\in M} \sum_{i=1}^{2^n} a_{n,i}h_{n,i} \right\|_X^2 \leqq C^2 \left\| \sum_{n\in M} \sum_{i=1}^{2^n} a_{n,i}f_{n,i} \right\|^2$$
$$\leqq C^2 \cdot \sum_{n\in M} \left\| \sum_{i=1}^{2^n} a_{n,i}f_{n,i} \right\|^2$$
$$\leqq C^4 \cdot \sum_{n\in M} \left\| \sum_{i=1}^{2^n} a_{n,i}h_{n,i} \right\|_X^2$$

Now for $l = 0, 1, 2, \cdots$ and $k = 1, \cdots, 2^l$, define

$$x_{l,k}(t) = \begin{cases} h_{n_{k+l},i}(t) & \text{if } t \in [(k-1)2^{-l}, k2^{-l}) \cap \operatorname{supp} h_{n_{k+l},i} \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence $(x_{l,k})_{k=1}^{2^l}$ has the same distribution as $(h_{l,k})_{k=1}^{2^l}$, and hence for any scalars $(a_k)_{k=1}^{2^l}$,

$$\left\| \sum_{k=1}^{2^l} a_k x_{l,k} \right\|_X = \left\| \sum_{k=1}^{2^l} a_k h_{l,k} \right\|_X,$$

while from (*) we have

$$\left\| \sum_{k=1}^{2^l} a_k x_{l,k} \right\|_X \overset{D}{\sim} \left(\sum_{k=1}^{2^l} | a_k |^2 \| x_{l,k} \|_X^2 \right)^{1/2}.$$

Thus, X is equal to $L_2[0,1]$ up to an equivalent renorming. □

REMARK 1. The Classification Theorem now yields that, for $w(x)$ regular and $1 < p < \infty$, $L_{w,p}[0,1]$ has unique r.i. structure on $[0,1]$. That is, if X is a r.i. function space on $[0,1]$ which is isomorphic to $L_{w,p}[0,1]$, then $X = L_{w,p}[0,1]$ up to an equivalent norm.

A major reduction in the proof of Theorem 1 entails comparing an arbitrary finite symmetric sequence in $L_{w,p}$ to a disjointly supported, equi-distributed sequence. The next lemma indicates the behavior of such a sequence when the weight $w(x)$ is submultiplicative (or, what is equivalent for $w(x)$ regular, when $S(x)$ is submultiplicative).

For $n = 1, 2, \cdots$ and $i = 1, \cdots, n$ we denote by $z_{n,i}$ the characteristic function of the interval $[(i-1)/n, i/n)$.

LEMMA 2. *Let $S(x)$ be submultiplicative with constant C and let $1 \leqq p < \infty$. If $(f_i)_{i=1}^n$ is a disjointly supported sequence in $L_{w,p}$ and each f_i has the same distribution, then*

$$\left\| \sum_{i=1}^n a_i f_i \right\| \leqq C^{1/p} \left\| \sum_{i=1}^n f_i \right\| \cdot \left\| \sum_{i=1}^n a_i z_{n,i} \right\|,$$

for every choice of scalars $(a_i)_{i=1}^n$.

PROOF. Since $(f_i)_{i=1}^n$ is a disjoint 1-symmetric sequence we have

$$\left\|\sum_{i=1}^n a_i f_i\right\| = \left\|\sum_{i=1}^n a_i^* f_i\right\| = \left\|\left(\sum_{i=1}^n a_i^{*p}|f_i|^p\right)^{1/p}\right\|$$

$$= \left\|\left\{\sum_{k=1}^n (a_k^{*p} - a_{k+1}^{*p})\left|\sum_{i=1}^k f_i\right|^p\right\}^{1/p}\right\|,$$

where $(a_i^*)_{i=1}^n$ is the decreasing rearrangement of $(|a_i|)_{i=1}^n$ and $a_{n+1}^* = 0$. And, since $L_{w,p}$ is p-convex with constant 1, it follows that

$$(**) \qquad \left\|\sum_{i=1}^n a_i f_i\right\| \leqq \left(\sum_{k=1}^n (a_k^{*p} - a_{k+1}^{*p})\left\|\sum_{i=1}^k f_i\right\|^p\right)^{1/p}.$$

Now, since $S(x)$ is submultiplicative, and since each f_i has the same distribution,

$$\left\|\sum_{i=1}^k f_i\right\|^p = \int_0^\infty S(kd_{f_1}(t))d(t^p)$$

$$= \int_0^\infty S\left(\frac{k}{n}\sum_{i=1}^n d_{f_i}(t)\right)d(t^p)$$

$$\leqq C\cdot S\left(\frac{k}{n}\right)\cdot\left\|\sum_{i=1}^n f_i\right\|^p.$$

Combining this calculation with (**) we have

$$\left\|\sum_{i=1}^n a_i f_i\right\| \leqq \left(\sum_{k=1}^n (a_k^{*p} - a_{k+1}^{*p})\left\|\sum_{i=1}^k f_i\right\|^p\right)^{1/p}$$

$$\leqq C^{1/p}\cdot\left\|\sum_{i=1}^n f_i\right\|\cdot\left(\sum_{k=1}^n (a_k^{*p} - a_{k+1}^{*p})S\left(\frac{k}{n}\right)\right)^{1/p}$$

$$= C^{1/p}\cdot\left\|\sum_{i=1}^n f_i\right\|\cdot\left\{\sum_{k=1}^n a_k^{*p}\cdot[S(k/n) - S((k-1)/n)]\right\}^{1/p}$$

$$= C^{1/p}\cdot\left\|\sum_{i=1}^n f_i\right\|\cdot\left\|\sum_{k=1}^n a_k z_{n,k}\right\|,$$

which completes the proof. □

REMARK 2. Note that (**) holds for any disjointly supported 1-symmetric sequence $(f_i)_{i=1}^n$ in $L_{w,p}$ for any $1 \leqq p < \infty$ and any weight $w(x)$.

The next lemma is a simple observation which will prove useful in the sequel:

LEMMA 3. *Let $(f_i)_{i=1}^n$ be disjointly supported in $L_{w,p}[0,\infty)$, $1 \leqq p < \infty$. Then,*

$$\left\|\sum_{i=1}^n f_i\right\|^p \geqq \frac{1}{n}\sum_{i=1}^n \|D_n f_i\|^p.$$

PROOF. Since $S(x)$ is concave,

$$\left\| \sum_{i=1}^{n} f_i \right\|^p = \int_0^\infty S\left(\sum_{i=1}^{n} d_{f_i}(t) \right) d(t^p)$$
$$\geqq \frac{1}{n} \sum_{i=1}^{n} \int_0^\infty S(n d_{f_i}(t)) d(t^p)$$
$$= \frac{1}{n} \sum_{i=1}^{n} \| D_n f_i \|^p. \qquad \square$$

We are now prepared to describe the process by which an arbitrary finite symmetric sequence in $L_{w,p}[0,1]$ may be compared to a disjointly supported, equi-distributed sequence. To this end we first require the notion of a symmetrically exchangeable sequence [4] (cf. section 1 and section 7 of [7] for applications of symmetrically exchangeable sequences in L_p-spaces and Orlicz spaces).

Let $(x_i)_{i=1}^n$ be a finite M-symmetric sequence in $L_{w,p}[0,1]$. Let Π_n be the set of all permutations of $\{1,\cdots,n\}$ and let $\{I_{\pi,\varepsilon} : \pi \in \Pi_n,\ \varepsilon \in \{-1,1\}^n\}$ be a partition of $[0,1]$ into mutually disjoint intervals, each of length $1/2^n n!$. Let $\psi_{\pi,\varepsilon} : I_{\pi,\varepsilon} \to [0,1]$ be the unique linear, increasing, onto map. For $1 \leqq i \leqq n$, define $y_i \in L_{w,p}[0,1]$ by

$$y_i(t) = \varepsilon_i x_{\pi(i)}(\psi_{\pi,\varepsilon}(t)) \qquad \text{for } t \in I_{\pi,\varepsilon}.$$

$(y_i)_{i=1}^n$ is a symmetrically exchangeable sequence; that is,

$$\operatorname{dist}(\varepsilon_1 y_{\pi(1)},\cdots,\varepsilon_n y_{\pi(n)}) = \operatorname{dist}(y_1,\cdots,y_n) \qquad \text{for any } \pi, \varepsilon.$$

In particular, each y_i has the same distribution and $(y_i)_{i=1}^n$ is a 1-symmetric sequence in $L_{w,p}[0,1]$.

It is shown in [4] and [7] that a finite M-symmetric sequence $(x_i)_{i=1}^n$ in L_p or in an Orlicz space is M-equivalent to the symmetrically exchangeable sequence $(y_i)_{i=1}^n$ obtained from $(x_i)_{i=1}^n$ by the above process. This is not the case even in L_{pq}, in general, as is demonstrated by example 10.7 of [7]. However, in the presence of a submultiplicative weight, a somewhat broader notion of equivalence is available. To see this, we will require further notation which will facilitate computations involving the square function $(\sum_{i=1}^n |a_i y_i|^2)^{1/2}$.

For $\pi \in \Pi_n$, define $J_\pi = \bigcup \{I_{\pi,\varepsilon} : \varepsilon \in \{-1,1\}^n\}$ and $\varphi_\pi : J_\pi \to [0,1]$ by $\varphi_\pi = \sum_\varepsilon \psi_{\pi,\varepsilon}$. Since $|J_\pi| = 1/n!$ and since $|\varphi_\pi^{-1}(A)| = |A|/n!$ for $A \subset [0,1]$, the distribution of $f(\varphi_\pi(t))$ is equal to $(1/n!)d_f$ for any measurable function f. In other words, if we define an operator D_π by

$$(D_\pi f)(t) = \begin{cases} f(\varphi_\pi(t)) & \text{if } t \in J_\pi, \\ 0 & \text{otherwise,} \end{cases}$$

then $D_\pi f$ and $D_{(1/n!)}f$ have the same distribution. Using this notation, a simple computation shows that $(\Sigma_{i=1}^n |a_i y_i|^2)^{1/2}$ may now be written as $\Sigma_\pi D_\pi(\Sigma_{i=1}^n |a_i x_{\pi(i)}|^2)^{1/2}$. In light of these remarks we have:

LEMMA 4. *Let $w(x)$ be regular and $1 \leqq p < \infty$. Given $M < \infty$, there exists a constant $C = C(M, w, p) < \infty$ such that if $(x_i)_{i=1}^n$ is a finite M-symmetric sequence in $L_{w,p}[0,1]$ and if $(y_i)_{i=1}^n$ is the symmetrically exchangeable sequence obtained from $(x_i)_{i=1}^n$ by the process described above, then*

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leqq C \left\| \sum_{i=1}^n a_i y_i \right\|$$

for every choice of scalars $(a_i)_{i=1}^n$.

PROOF. Since $w(x)$ is regular, $L_{w,p}[0,1]$ is q-concave for some $q < \infty$, and hence by [8, theorem 1.d.6] there exists a constant $C_1 = C_1(w, p) < \infty$ such that

$$\frac{1}{\sqrt{2}} \left\| \left(\sum_{i=1}^k |f_i|^2 \right)^{1/2} \right\| \leqq \int_0^1 \left\| \sum_{i=1}^k r_i(t) f_i \right\| dt \leqq C_1 \cdot \left\| \left(\sum_{i=1}^k |f_i|^2 \right)^{1/2} \right\|$$

for any $(f_i)_{i=1}^k$ in $L_{w,p}[0,1]$, where $(r_i)_{i=1}^k$ are the Rademacher functions on $[0,1]$. Thus,

$$\begin{aligned} \left\| \sum_{i=1}^n a_i y_i \right\| &\geqq \frac{1}{\sqrt{2}} \left\| \left(\sum_{i=1}^n |a_i y_i|^2 \right)^{1/2} \right\| \\ &= \frac{1}{\sqrt{2}} \left\| \sum_\pi D_\pi \left(\sum_{i=1}^n |a_i x_{\pi(i)}|^2 \right)^{1/2} \right\|. \end{aligned}$$

Hence, by Lemma 3 and by the fact that $(x_i)_{i=1}^n$ is M-symmetric, we get

$$\begin{aligned} \left\| \sum_{i=1}^n a_i y_i \right\| &\geqq \frac{1}{\sqrt{2}} \left(\frac{1}{n!} \sum_\pi \left\| \left(\sum_{i=1}^n |a_i x_{\pi(i)}|^2 \right)^{1/2} \right\|^p \right)^{1/p} \\ &\geqq \frac{1}{\sqrt{2}} C_1^{-1} \cdot M^{-1} \cdot \left(\frac{1}{n!} \sum_\pi \left\| \sum_{i=1}^n a_i x_{\pi(i)} \right\|^p \right)^{1/p} \\ &\geqq \frac{1}{\sqrt{2}} \cdot C_1^{-1} \cdot M^{-2} \cdot \left\| \sum_{i=1}^n a_i x_i \right\|. \end{aligned}$$

□

REMARK 3. It is not difficult to see that $\|(\Sigma_{i=1}^n |y_i|^2)^{1/2}\| = \|(\Sigma_{i=1}^n |x_i|^2)^{1/2}\|$, and hence

$$\begin{aligned} \left\| \sum_{i=1}^n y_i \right\| &\leqq C_1 \left\| \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2} \right\| = C_1 \cdot \left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\| \\ &\leqq \sqrt{2} \cdot C_1 \cdot M \cdot \left\| \sum_{i=1}^n x_i \right\|. \end{aligned}$$

Our last lemma shows that the conclusion of Lemma 2 still holds for symmetrically exchangeable sequences which are not necessarily disjoint. It is convenient in what follows to have $w(x)$ defined on all of $(0,\infty)$. If $w(x)$ is regular and submultiplicative, a simple calculation shows that by defining $\tilde{w}(x)=w(x)$ if $x \leqq 1$ and $\tilde{w}(x)=w(1)$ for $x>1$, we may assume $w(x)$ is regular and submultiplicative on $(0,\infty)$.

LEMMA 5. *Let $w(x)$ be regular and submultiplicative and let $2 \leqq p < \infty$. There exists a constant $C = C(w,p) < \infty$ such that if $(y_i)_{i=1}^n$ is a symmetrically exchangeable sequence in $L_{w,p}[0,1]$, then*

$$\left\| \sum_{i=1}^n a_i y_i \right\| \leqq C \cdot \left\| \sum_{i=1}^n y_i \right\| \cdot \left\| \sum_{i=1}^n a_i z_{n,i} \right\|$$

for every choice of scalars $(a_i)_{i=1}^n$.

PROOF. Let $(\tilde{y}_i)_{i=1}^n$ be a disjointly supported sequence in $L_{w,p}[0,\infty)$ with $d_{\tilde{y}_i} = d_{y_i}$, $i=1,\cdots,n$. Then, by Lemma 2, there is a constant $C_2 = C_2(w,p) < \infty$ so that

$$\left\| \sum_{i=1}^n a_i \tilde{y}_i \right\| \leqq C_2 \cdot \left\| \sum_{i=1}^n \tilde{y}_i \right\| \cdot \left\| \sum_{i=1}^n a_i z_{n,i} \right\|,$$

and by corollary 7.3 of [7],

$$\left\| \sum_{i=1}^n \tilde{y}_i \right\| \leqq \sqrt{2} \cdot \left\| \sum_{i=1}^n y_i \right\|.$$

Using these observations and the fact that $(y_i)_{i=1}^n$ is symmetrically exchangeable, the Classification Formula simplifies substantially; in particular, there is a constant $C_3 = C_3(w,p) < \infty$ so that

$$\begin{aligned}
\left\| \sum_{i=1}^n a_i y_i \right\| &\leqq C_3 \cdot \max\left\{ \left\| \max_{1 \leqq i \leqq n} |a_i y_i| \right\|, \left\| \sum_{i=1}^n y_i \right\| \cdot \left(\sum_{i=1}^n |a_i|^2/n \right)^{1/2} \right\} \\
&\leqq C_3 \cdot \max\left\{ \left\| \sum_{i=1}^n a_i \tilde{y}_i \right\|, \left\| \sum_{i=1}^n y_i \right\| \cdot \left(\sum_{i=1}^n |a_i|^2/n \right)^{1/2} \right\} \\
&\leqq C_2 \cdot C_3 \cdot \sqrt{2} \cdot \left\| \sum_{i=1}^n y_i \right\| \cdot \max\left\{ \left\| \sum_{i=1}^n a_i z_{n,i} \right\|, \left\| \sum_{i=1}^n a_i z_{n,i} \right\|_{L_2} \right\} \\
&= C_2 \cdot C_3 \cdot \sqrt{2} \cdot \left\| \sum_{i=1}^n y_i \right\| \cdot \left\| \sum_{i=1}^n a_i z_{n,i} \right\|,
\end{aligned}$$

the last inequality following from the fact that $\|f\| \geqq \|f\|_{L_2}$ for $f \in L_{w,p}$ $(p \geqq 2)$. □

As a consequence of the preceding remarks we now have:

THEOREM 1. *Let $w(x)$ be regular and submultiplicative, and let $2 \leqq p < \infty$. Let X be a* r.i. *function space on $[0,1]$ which is isomorphic to a subspace of $L_{w,p}[0,1]$. Then, up to an equivalent norm, $X = L_{w,p}[0,1]$ or $X = L_2[0,1]$.*

PROOF. By the Classification Theorem and the remarks thereafter, if X is not isomorphic to $L_2[0,1]$, then there exists a constant $C_1 < \infty$ such that $\|f\| \leqq C_1 \cdot \|f\|_X$ for all $f \in X$.

Let T be an isomorphism from X into $L_{w,p}[0,1]$ and set $x_{n,i} = Tz_{n,i}$, $n = 1,2,\cdots, i = 1,\cdots,n$. If $M = \|T\| \cdot \|T^{-1}\|$, then for each n, $(x_{n,i})_{i=1}^n$ is M-symmetric in $L_{w,p}[0,1]$ with $\|\sum_{i=1}^n x_{n,i}\| \leqq \|T\|$.

For each n, let $(y_{n,i})_{i=1}^n$ be the symmetrically exchangeable sequence derived from $(x_{n,i})_{i=1}^n$ in the manner described preceding Lemma 4. By Lemma 4 and Lemma 5, there is a constant $C_2 = C_2(T, w, p) < \infty$ such that for all n and scalars $(a_i)_{i=1}^n$ we have

$$\left\| \sum_{i=1}^n a_i x_{n,i} \right\| \leqq C_2 \cdot \left\| \sum_{i=1}^n y_{n,i} \right\| \cdot \left\| \sum_{i=1}^n a_i z_{n,i} \right\|.$$

Now $\|\sum_{i=1}^n a_i z_{n,i}\|_X \leqq \|T^{-1}\| \cdot \|\sum_{i=1}^n a_i x_{n,i}\|$, and by Remark 3, $\|\sum_{i=1}^n y_{n,i}\| \leqq C_2 \|\sum_{i=1}^n x_{n,i}\| \leqq C_2 \|T\|$; hence, there is a constant $C_3 = C_3(T, w, p) < \infty$ such that

$$\left\| \sum_{i=1}^n a_i z_{n,i} \right\|_X \leqq C_3 \cdot \left\| \sum_{i=1}^n a_i z_{n,i} \right\|$$

for all n and scalars $(a_i)_{i=1}^n$.

Since the dyadic simple functions are dense in $L_{w,p}[0,1]$, we have shown that

$$C_1^{-1} \cdot \|f\| \leqq \|f\|_X \leqq C_3 \cdot \|f\| \qquad \text{for all } f \in X,$$

which completes the proof. □

In order to prove the converse of Theorem 1, we need only examine a certain class of sublattices of $L_{w,p}[0,1]$ (cf. section 7 of [7]). Given measurable functions f and g on $[0,1]$, define $f \otimes g$ on $[0,1]^2$ by $(f \otimes g)(s,t) = f(s) \cdot g(t)$. Since $(f \otimes g)^*$ is again measurable on $[0,1]$, we may define $\|f \otimes g\| = \|(f \otimes g)^*\|$. If $g \in L_{w,p}[0,1]$ is fixed, it is possible to define a r.i. function space X_g on $[0,1]$ which is also a closed sublattice of $L_{w,p}[0,1]$. Indeed, the space X_g is defined to be the completion of the integrable simple functions under the norm $\|f\|_{X_g} = \|f \otimes g\|$. To see that X_g is closed observe that since $L_{w,p}[0,1]$ is p-convex:

$$\|f(s)\otimes g(t)\| = \left(\int_0^1 \|f(s)\otimes g(t+u)\|^p du\right)^{1/p}$$

$$\geqq \left\| \left(\int_0^1 |f(s)\otimes g(t+u)|^p du\right)^{1/p} \right\| \quad \text{(where } t+u \text{ is taken mod 1)}$$

$$= \|f\|\cdot\|g\|_{L_p}.$$

With this notation it is now possible to give a simple proof of:

THEOREM 2. *Let $w(x)$ be regular and $1<p<\infty$. If X_g is isomorphic to $L_{w,p}[0,1]$ for all $g\in L_{w,p}[0,1]$, then $S(x)$ is submultiplicative.*

PROOF. By Remark 1, if X_g is isomorphic to $L_{w,p}[0,1]$ for $\|g\|=1$, then there exists a constant $C_g<\infty$ such that $\|f\otimes g\| = \|f\|_{X_g} \leqq C_g\cdot\|f\|$ for all $f\in L_{w,p}[0,1]$. Now, since the mapping $f\to f\otimes g$ is linear, the Uniform Boundedness Principle gives a constant $C<\infty$ such that $\|f\otimes g\|\leqq C\|f\|\|g\|$ for all $f,g\in L_{w,p}[0,1]$. In particular, if $f=\chi_{[0,x]}$, $g=\chi_{[0,y]}$, $0\leqq x,y\leqq 1$, we have:

$$S(xy) = \|f\otimes g\|^p \leqq C^p\|f\|^p\|g\|^p = C^pS(x)S(y),$$

which completes the proof. □

3. Finally, we present an example (adapted from [2]) which demonstrates the difficulty in classifying the subspaces X_g of $L_{w,p}[0,1]$ if $w(x)$ is not submultiplicative.

EXAMPLE. Given $1<p<\infty$, there is a Lorentz function space $L_{w,p}[0,1]$ having a closed sublattice which is not isomorphic to any Lorentz function space $L_{v,p}[0,1]$.

PROOF. Let $1<p<\infty$ and let $w(x)=x^{-1/2}(1-\log x)^{-2}$. Let $g(x)=x^{-1/2p}$. Then $g\in L_{w,p}[0,1]$, since

$$\int_0^1 g(t)^p w(t)dt = \int_0^1 t^{-1}(1-\log t)^{-2}dt = 1.$$

Now, since $w(x)$ is regular, X_g is superreflexive. Hence, if X_g were isomorphic to some $L_{v,p}[0,1]$, then $v(x)$ must be regular and by Remark 1 we must have $X_g = L_{v,p}[0,1]$ up to an equivalent norm. In particular, we have

$$xv(x)\sim\|\chi_{(0,x)}\|^p_{L_{v,p}}\sim\|\chi_{(0,x)}\|^p_{X_g}$$

$$=\int_0^x g(t/x)^p w(t)dt.$$

That is, $v(x) \sim x^{-1}\int_0^x g(t/x)^p w(t)dt = x^{-1/2}(1-\log x)^{-1}$. Now a simple calculation yields that

$$(g \otimes g)^*(t) \sim t^{-1/2p}(1-\log t)^{1/2p},$$

and thus $g \in X_g$ since

$$\int_0^1 (g \otimes g)^*(t)^p w(t)dt \sim \int_0^1 t^{-1}(1-\log t)^{-3/2}dt = 2,$$

but $g \notin L_{v,p}[0,1]$ since

$$\int_0^1 g(t)^p v(t)dt \sim \int_0^1 t^{-1}(1-\log t)^{-1}dt = \infty.$$

This contradiction completes the proof. □

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